JAMES RUSE A H S

GESTA NON VERBA

TRIAL HIGHER SCHOOL CERTIFICATE EXAMINATION 2008

MATHEMATICS EXTENSION 2

Time Allowed – 3 Hours
(Plus 5 minutes Reading Time)

- *All* questions may be attempted
- *All* questions are of equal value
- Department of Education approved calculators are permitted
- In every question, show all necessary working
- Marks may not be awarded for careless or badly arranged work
- No grid paper is to be used unless provided with the examination paper

The answers to all questions are to be returned in separate *stapled* bundles clearly labeled Question 1, Question 2, etc. Each question must show your Candidate Number.

QUESTION 1 Start a new Page

Marks

(a) Evaluate $\int_1^3 \frac{1}{x^2-4x+5}\,dx$. **2**

(b) Let $I_n = \int_0^\pi x^n \sin x\, dx$, where $n = 0, 1, 2, \ldots\ldots$

(i) Show $I_n = \pi^n - n(n-1)I_{n-2}$ for $n = 2, 3, 4, \ldots\ldots.$ **3**

(ii) Hence, evaluate $\int_0^\pi x^4 \sin x dx$. **2**

(c) (i) Let $\frac{1}{x(\pi-2x)} = \frac{A}{x} + \frac{B}{\pi-2x}$. **1**

Find the real values for A and B.

(ii) Hence or otherwise, show $\int_{\frac{\pi}{6}}^{\frac{\pi}{3}} \frac{dx}{x(\pi-2x)} = \frac{2}{\pi}\ln 2$. **2**

(iii) Using the substitution $u = a+b-x$, show that $\int_a^b f(x)\,dx = \int_a^b f(a+b-x)\,dx$. **2**

(iv) Evaluate $\int_{\frac{\pi}{6}}^{\frac{\pi}{3}} \frac{\cos^2 x}{x(\pi-2x)}\,dx$. **3**

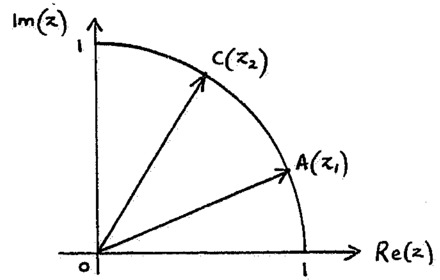
QUESTION 2 Start a new Page

Marks

(a) Express $z = \frac{7+4i}{3-2i}$ in the form $a+ib$, where a and b are real. **1**

(b) On an Argand diagram sketch the locus of the points representing the complex number z where $|z-3-i| = \sqrt{10}$.
Hence, find the greatest value of $|z|$ subject to this condition. **3**

(c)



In the Argand diagram above, the two points A and C lie on the circumference of the circle with centre the origin of radius 1. They represent the complex numbers z_1 and z_2 respectively.

(i) Copy the diagram into your answer sheet. Mark on your diagram the position of the point B that represents the complex number $z_1 + z_2$. **1**

(ii) Explain why AC is perpendicular to OB. **1**

(d) For the complex number $z = x + iy$:

(i) Find the equation of the curve in the Argand diagram for which $\text{Re}(z^2) = 3$, and sketch the curve showing any intercepts and asymptotes. **2**

(ii) Find the equation of the curve such that $\text{Im}(z^2) = 4$. **1**

(iii) Hence, or otherwise, solve the equation $z^2 = 3 + 4i$. **3**

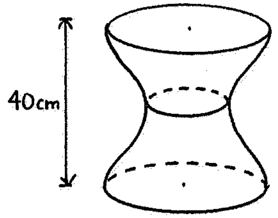
(iv) The region R in the Argand diagram consists of the set of all values of z such that $0 < \text{Re}(z^2) < 3$ and $0 < \text{Im}(z^2) < 4$. **3**

Draw a sketch of the region R, indicating the coordinates of the intercepts.

QUESTION 3 Start a new Page

Marks

(a) A stool of height 40 cm has the shape of two identical truncated cones with curved sides as shown in the diagram below. **3**



For the bottom half of the stool, the cross-section at height h above the ground is a circle parallel to the base and of radius $r(h)$, where

$$r(h)=\frac{75\sqrt{2}}{\sqrt{h^2+50}}$$ DO NOT PROVE THIS FORMULA

Find the total volume of the stool to the nearest cubic centimetre.

(b) The area bounded by $y=4-x^2$, $x=2$ and $y=4$ is rotated about the line $x=4$. Using the method of cylindrical shells:

(i) Show that the volume of a cylindrical shell of thickness δx is $\pi x^2(8-2x)\delta x$. **2**

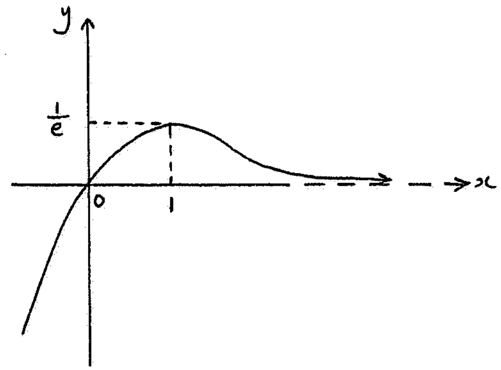
(ii) Find the volume of the solid generated. **2**

QUESTION 3 CONTINUES OVER PAGE

QUESTION 3 CONTINUED

Marks

(c) The graph of $y=xe^{-x}$ is sketched below:



On separate axes, sketch the following curves. Indicate clearly any turning points, asymptotes, and intercepts with the coordinate axes.

(i) $y=x^2e^{-2x}$ **2**

(ii) $y=\frac{1}{x^2e^{-2x}}$ **2**

(iii) $y=\log_e\left(xe^{-x}\right)$ **2**

(iv) $y=e^{xe^{-x}}$ **2**

QUESTION 4 Start a new Page

Marks

(a) Let $P(z)=z^7-1$.

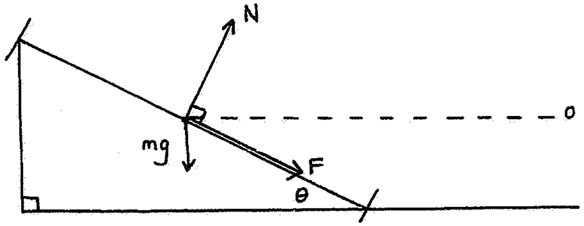
(i) Find all the complex roots of $P(z)=0$. Let these roots be $z_0, z_1, \ldots\ldots$ and z_6 leaving all answers in mod-arg form. **2**

(ii) Plot the points representing $z_0, z_1, \ldots\ldots$ and z_6 on the Argand diagram. **2**

(iii) Factorize $P(z)$ over the set of real numbers. **3**

(iv) Hence, or otherwise, show that $\cos\frac{2\pi}{7}+\cos\frac{4\pi}{7}+\cos\frac{6\pi}{7}=-\frac{1}{2}$. **2**

b)



The railway line which moves around a circular track of mean radius 800 metres, is banked by raising the outer rail to a certain level above the inner rail.

(i) When the train travels at 20 m/s the lateral thrust, F is on the outer rail. Show that $F=m\left(\frac{1}{2}\cos\theta-g\sin\theta\right)$. **2**

(ii) When the train travels at 10 m/s, the lateral thrust on the inner rail is the same as the lateral thrust on the outer rail at a speed of 20m/s.

(α) Find the angle of the banking. **2**

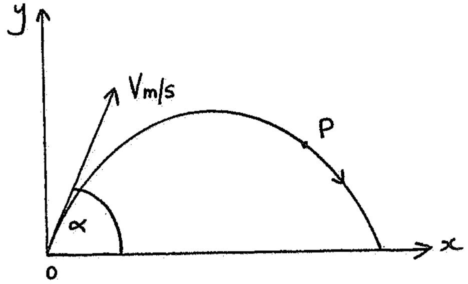
(β) Find the speed of the train when there is no lateral thrust exerted on the rails. **2**

Use $g=9.8\ \text{ms}^{-2}$.

QUESTION 5 Start a new Page

Marks

A particle P is projected from a point O at ground level with a speed V m/s at an angle of elevation $\alpha\,^o$ as shown below.



Given that the equations of motion for the particle P at time t seconds is given by (air resistance is neglected):

$x=Vt\cos\alpha$ and $y=-\frac{gt^2}{2}+Vt\sin\alpha$ DO NOT PROVE THESE EQUATIONS.

where g is the acceleration due to gravity and g is measured in ms^{-2}.

(a) If the highest point of the trajectory of the particle P has coordinates (C, H)

(i) Show that the angle of projection is $\tan^{-1}\frac{2H}{C}$. **4**

(ii) Show that the speed of projection is given by $V^2=\frac{g}{2H}\left(4H^2+C^2\right)$. **3**

(b) At the same time that particle P is projected, a second particle Q is projected horizontally with speed U m/s from a point at height h metres vertically above O, so that the particles move in the same vertical plane.

(i) Show that if the particles collide, then $V>U$. **3**

(ii) Find the time at which collision takes place, in terms of h, V and U. **2**

(iii) Show that, if the particles collide at ground level, then **3**

$$V^2=U^2+\frac{1}{2}gh.$$

QUESTION 6 Start a new Page

Marks

(a) $P(x)$ is a cubic polynomial with real coefficients. **4**
One zero of $P(x)$ is $1+2i$, the constant term is -15 and $P(2)=5$.

Write $P(x)$ with real coefficients.

(b) The equation $x^3-4x+5=0$ has roots α, β and γ.

(i) Find the value of $\alpha^3+\beta^3+\gamma^3$. **2**

(ii) Find the value of $(\alpha+\beta)^2(\alpha+\gamma)^2(\beta+\gamma)^2$. **2**

(c) A straight line is drawn to the curve $y=x^4-4x^3-18x^2$ so that it is a common tangent at two distinct points on the curve.

(i) If the equation of the tangent is $y=mx+b$, where $b<0$, and its points of contact are $x=p$ and $x=q$,

(α) Show that $p+q=2$; **2**

(β) Show that $p^2q^2=-b$. **1**

(ii) Hence, or otherwise, find the equation of the common tangent. **4**

QUESTION 7 Start a new Page

Marks

(a) A magic square is shown below:

4	3	8
9	5	1
2	7	6

Note that the sum of the diagonals, rows and columns is 15.

Three different numbers are chosen at random from the square.
Find the probability that the sum of the numbers is 15, if:

(i) A 5 is chosen first. **1**

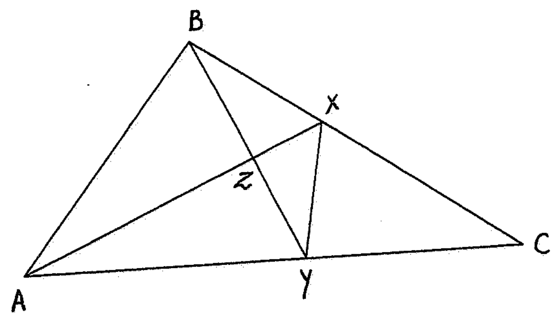
(ii) A 2 is chosen first. **2**

(b) X and Y are points on the sides BC and AC of a triangle ABC respectively such that $\angle AXC=\angle BYC$ and $BX=XY$.

Copy the diagram onto your examination paper then,

(i) Prove $ABXY$ is a cyclic quadrilateral. **2**

(ii) Hence or otherwise, prove AX bisects $\angle BAC$. **2**



QUESTION 7 CONTINUES OVER PAGE

QUESTION 7 CONTINUED

Marks

(c) A particle of mass 10 kg is found to experience a resistive force, in Newtons, of one-ninth of the square of its velocity v, in metres per second, when it moves through the air.

The particle is projected vertically upwards from a point O with a velocity of $30\sqrt{3}$ m/s and the point A, vertically above O, is the highest point reached by the particle before it starts to fall to the ground again.

Assuming the value of $g = 10\text{ ms}^{-2}$

(i) Explain why $\ddot{x} = -10 - \frac{1}{90}v^2$. **1**

(ii) Find the time the particle takes to reach A from O. **2**

(d) (i) Show that: **2**

$$\cot 2x - \tan 2x = 2\cot 4x.$$

(ii) Hence, prove by mathematical induction that for $n = 1, 2, 3......$ **3**

$$\tan x + 2\tan 2x + 4\tan 4x + ...+ 2^{n-1}\tan(2^{n-1}x) = \cot x - 2^n \cot(2^n x).$$

QUESTION 8 Start a new Page

Marks

(a) (i) Show that: **2**

$$(1+x)^{2n} + (1-x)^{2n} = 2\sum_{r=0}^{n} {}^{2n}C_{2r}\, x^{2r} \text{ for n = 1, 2, 3, ...}$$

(ii) An alphabet only consists of three letters A, B and C.

(α) Explain why the number of words consisting of five letters containing exactly 2 A's is given by ${}^5C_2 \times 2^3$. **2**

(β) Show that the number of words consisting of $2n$ letters having zero or an even number of A's, is given by: **3**

$$\frac{1}{2}\left(3^{2n}+1\right).$$

(b) (i) Show that the normal at the point $P\left(cp, \frac{c}{p}\right)$ to the hyperbola $xy = c^2$ is given by **2**

$$p^3x - py = c\left(p^4 - 1\right).$$

(ii) If this normal meets the hyperbola again at $Q\left(cq, \frac{c}{q}\right)$, show that **2**

$$p^3 q = -1.$$

(iii) Hence, find the area of the triangle PQR, where R is the point of intersection of the tangent at P with the y-axis. **2**

You may assume that the equation of the tangent is given by $x + p^2 y = 2cp$.

(iv) What is the value(s) of p that produces a triangle of minimum area? **2**

This is the end of the exam

Question 1 (15 Marks)

(a) $\int_1^3 \frac{1}{x^2-4x+5}\,dx = \int_1^3 \frac{1}{x^2-4x+4+1}\,dx$

$= \int_1^3 \frac{1}{(x-2)^2+1}\,dx$ 1

$= \left[\tan^{-1}(x-2)\right]_1^3$

$= \tan^{-1}1 - \tan^{-1}(-1)$

$= 2\tan^{-1}1$

$= 2\times\frac{\pi}{4}$

$= \frac{\pi}{2}$ $\boxed{2}$ 1

(c) $\int_{\pi/6}^{\pi/3} \frac{dx}{x(\pi-2x)}$

(i) Let $\frac{1}{x(\pi-2x)} = \frac{A}{x} + \frac{B}{\pi-2x}$

$1 = A(\pi-2x) + Bx$

$1 = (B-2A)x + A\pi$

$\therefore A = \frac{1}{\pi}, \quad B = \frac{2}{\pi}$ $\boxed{1}$ 1

(ii) $\therefore \int_{\frac{\pi}{6}}^{\frac{\pi}{3}} \frac{dx}{x(\pi-2x)} = \int_{\frac{\pi}{6}}^{\frac{\pi}{3}} \frac{1/\pi}{x} + \frac{2/\pi}{\pi-2x}\,dx$

$= \frac{1}{\pi}\int_{\frac{\pi}{6}}^{\frac{\pi}{3}} \frac{1}{x} + \frac{2}{\pi-2x}\,dx$

$= \frac{1}{\pi}\left[\ln x - \ln(\pi-2x)\right]_{\pi/6}^{\pi/3}$ 1

$= \frac{1}{\pi}\left[\ln\frac{x}{\pi-2x}\right]_{\pi/6}^{\pi/3}$

$= \frac{1}{\pi}\left(\ln\left(\frac{\pi/3}{\pi/3}\times\frac{2\pi/3}{\pi/6}\right)\right)$ 1

$= \frac{1}{\pi}\ln 4$

$= \frac{2}{\pi}\ln 2$ $\boxed{2}$

(iii) Let $u = a+b-x$

$du = -dx$

If $x=a \Rightarrow u=b$

$x=b \Rightarrow u=a$

$\therefore \int_a^b f(a+b-x)\,dx = \int_b^a f(u)\cdot(-1)\,du$ 1

$= \int_a^b f(u)\,du$ 1

Change variable $u \Rightarrow x$

$= \int_a^b f(x)\,dx$. $\boxed{2}$

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(iv) $\int_{\frac{\pi}{6}}^{\frac{\pi}{3}} \frac{\cos^2 x}{x(\pi-2x)}\,dx$

$= \int_{\frac{\pi}{6}}^{\pi/3} \frac{\cos^2\left(\frac{\pi}{6}+\frac{\pi}{3}-x\right)}{\left(\frac{\pi}{6}+\frac{\pi}{3}-x\right)\left(\pi-2\left(\frac{\pi}{6}+\frac{\pi}{3}-x\right)\right)}\,dx$

$= \int_{\pi/6}^{\pi/3} \frac{\cos^2\left(\frac{\pi}{2}-x\right)}{\left(\frac{\pi}{2}-x\right)\left(\pi-2\left(\frac{\pi}{2}-x\right)\right)}\,dx$

$= \int_{\frac{\pi}{6}}^{\pi/3} \frac{\sin^2 x}{\left(\frac{\pi}{2}-x\right)(2x)}\,dx$ 1

$= \int_{\frac{\pi}{6}}^{\pi/3} \frac{\sin^2 x}{(\pi-2x)x}\,dx$

$= \int_{\frac{\pi}{6}}^{\pi/3} \frac{1}{(\pi-2x)x}\,dx - \frac{\cos^2 x}{(\pi-2x)x}\,dx$

Let $I = \int_{\frac{\pi}{6}}^{\pi/3} \frac{\cos^2 x}{x(\pi-2x)}\,dx$, then

$I = \int_{\frac{\pi}{6}}^{\pi/3} \frac{1}{(\pi-2x)x}\,dx - I$ 1

$2I = \frac{2}{\pi}\ln 2$

$\therefore I = \frac{1}{\pi}\ln 2$ $\boxed{3}$ 1

(b) $I_n = \int_0^\pi x^n \sin x \, dx$

(i) $= \int_0^\pi x^n \cdot \frac{d}{dx}(-\cos x) \, dx$

$= \left[-x^n \cos x\right]_0^\pi - \int -\cos x \cdot n x^{n-1} \, dx$ 1

$u = x^n$, $\frac{du}{dx} = n x^{n-1}$, $\frac{dv}{dx} = \sin x$, $v = -\cos x$

$= -\pi^n \cos\pi - 0 + n \int x^{n-1} \cos x \, dx$

$= \pi^n + n\left[x^{n-1} \cdot \sin x\right]_0^\pi - n \int (n-1) x^{n-2} \cdot \sin x \, dx$ 1

$u = x^{n-1}$, $\frac{du}{dx} = (n-1) x^{n-2}$, $\frac{dv}{dx} = \cos x$, $v = \sin x$

$= \pi^n \cancel{+0} - n(n-1) \int x^{n-2} \sin x \, dx$ 1

$\therefore I_n = \pi^n - n(n-1) \cdot I_{n-2}$ $\boxed{3}$

(ii) $\int_0^\pi x^4 \sin x \, dx = \pi^4 - 4(3) I_2$

$= \pi^4 - 12\left[\pi^2 - 2(1) I_0\right]$

$= \pi^4 - 12\pi^2 + 24 I_0$ 1

$I_0 = \int_0^\pi x^0 \sin x \, dx$

$= \int_0^\pi \sin x \, dx$

$= \left[-\cos x\right]_0^\pi$

$= -\cos\pi + \cos 0$

$= 1 + 1$

$= 2$

$\therefore \int_0^\pi x^4 \sin x \, dx = \underline{\pi^4 - 12\pi^2 + 48}$ $\boxed{2}$ 1

Question 2 (15 Marks)

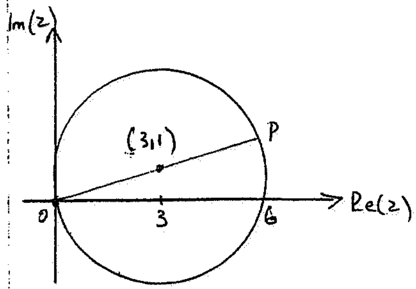
(a) $\frac{7+4i}{3-2i} \times \frac{3+2i}{3+2i} = \frac{21 + 14i + 12i - 8}{9 + 4}$

$= \frac{13 + 26i}{13}$

$= \underline{1 + 2i}$ $\boxed{1}$ 1

(b) $|z - 3 - i| = \sqrt{10}$

$|z - (3+i)| = \sqrt{10}$ is a circle centre $(3, 1)$, radius $\sqrt{10}$ 1

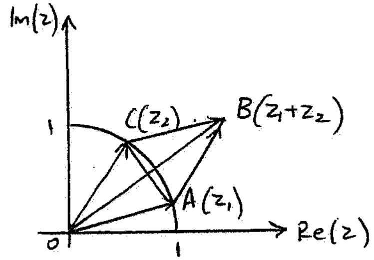


Greatest value of $|z|$ is OP 1

$= \underline{2\sqrt{10}}$ 1

$\boxed{3}$

(c) (i)



$\boxed{1}$ 1

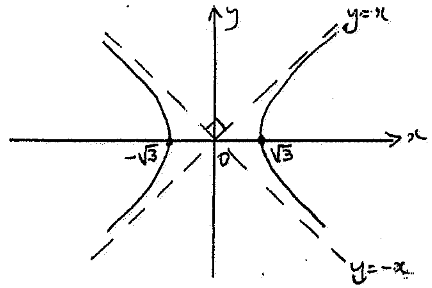
(ii) $AC \perp OB$ as OABC is a <u>rhombus</u>, since <u>all</u> sides are equal and diagonals intersect at right angles $\boxed{1}$ 1 (d)(i) $z^2 = (x+iy)^2 = x^2 - y^2 + 2ixy$

$Re(z^2) = 3$

$Re(x^2 - y^2 + 2ixy) = 3$

$\therefore x^2 - y^2 = 3$ OR 1

$\frac{x^2}{3} - \frac{y^2}{3} = 1$



1

$\boxed{2}$

(ii) $Im(z^2) = 4$

$Im(x^2 - y^2 + 2ixy) = 4$

$2xy = 4$

$\underline{xy = 2}$ 1

$\boxed{1}$

(iii) If $z^2 = 3+4i$ then $Re(z^2) = 3$ and $Im(z^2) = 4$

So the solutions of $z^2 = 3+4i$ are the points of intersection of the curves $x^2 - y^2 = 3$ & $xy = 2$.

$x^2 - y^2 = 3$

$xy = 2 \rightarrow y = \frac{2}{x}$

$x^2 - \left(\frac{2}{x}\right)^2 = 3$

$x^4 - 4 = 3x^2$

$x^4 - 3x^2 - 4 = 0$

$(x^2 - 4)(x^2 + 1) = 0$

$x = \pm 2$ only as $x^2 + 1 \neq 0$ since x must be real. 1 + 1

$\therefore$ Pts of intersection are $(2,1)$ and $(-2,-1)$

So the solutions to $z^2 = 3+4i$ are

$\underline{z = 2+i}$ or $\underline{-2-i}$ 1

$\boxed{3}$

Alternative Solution to (iii)

$z^2 = 3+4i$

Let $z^2 = r(\cos\theta + i\sin\theta)$ where $r = \sqrt{3^2+4^2} = 5$

$= 5(\cos\theta + i\sin\theta)$

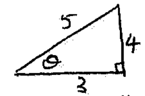
$z = \pm\sqrt{5}(\cos\theta + i\sin\theta)^{1/2}$

$= \pm\sqrt{5}\left(\cos\frac{\theta}{2} + i\sin\frac{\theta}{2}\right)$

$= \pm\sqrt{5}\left(\frac{2}{\sqrt{5}} + i\frac{1}{\sqrt{5}}\right)$ *

$\underline{\underline{z = \pm(2+i)}}$

Now



$\sin\theta = \frac{4}{5}$

$\cos\theta = \frac{3}{5}$

$\sin^2\frac{\theta}{2} = \frac{1}{2}(1-\cos\theta)$

$= \frac{1}{2}(1 - 3/5)$

$= \frac{1}{5}$

$\sin\frac{\theta}{2} = +\frac{1}{\sqrt{5}}$ only *

as $\frac{\theta}{2}$ is acute

Also $\cos^2\frac{\theta}{2} = \frac{1}{2}(1+\cos\theta)$

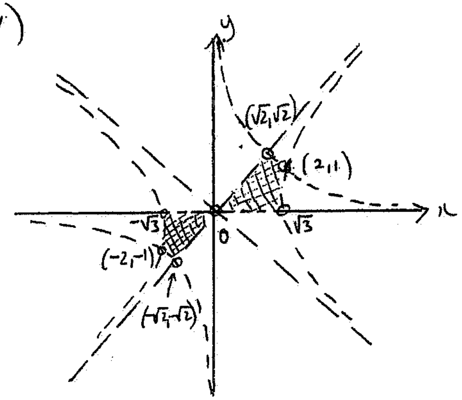
$= \frac{1}{2}\left(1+\frac{3}{5}\right)$

$= \frac{4}{5}$

$\cos\frac{\theta}{2} = +\frac{2}{\sqrt{5}}$ only *

as $\frac{\theta}{2}$ is acute

(iv)



NB: No boundary points are included in any of the regions.

All boundary lines are dotted. $\boxed{3}$

1+1 for eac[h] regio[n]

1 for do[tted] bounda[ry]

$\left.\begin{array}{l} xy = 2 \\ y = x \end{array}\right]$

$x^2 = 2$

$x = \pm\sqrt{2}$

Question 3 (15 Marks)

(a) $V = 2\int_0^{20} \pi r^2 \, dh$

$= 2\pi \int_0^{20} \left(\dfrac{75\sqrt{2}}{\sqrt{h^2+50}}\right)^2 dh$

$= 2\pi \int_0^{20} \dfrac{75^2 \cdot 2}{h^2+50} \, dh$ — 1

$= 4\pi(75^2)\int_0^{20} \dfrac{1}{h^2+50} \, dh$

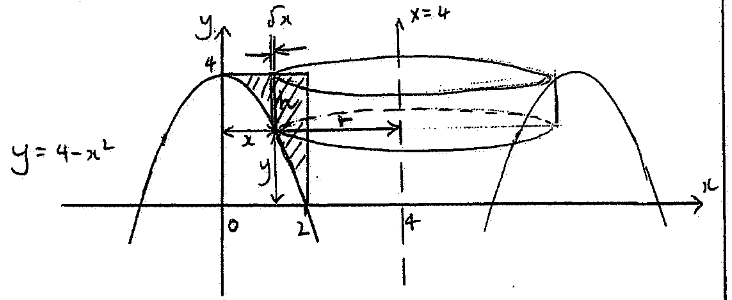
$= 4\pi(75^2) \dfrac{1}{\sqrt{50}}\left[\tan^{-1}\dfrac{h}{\sqrt{50}}\right]_0^{20}$ — 1

$= \dfrac{4\pi(75^2)}{\sqrt{50}} \tan^{-1}\dfrac{20}{\sqrt{50}}$

$= 12305{\cdot}26933 \text{ cm}^3$

$= 12305 \text{ cm}^3$ (nearest cubic centimetre). $\boxed{3}$ — 1

(b)



(i) $r = 4 - x$ $\qquad h = 4 - y$

$\qquad = 4 - (4 - x^2)$

$\therefore h = x^2$ — 1 for radius & height

$\delta V = 2\pi r h \, \delta x$

$= 2\pi(4-x)x^2 \, \delta x$

$\therefore \delta V = \underline{\pi x^2 (8 - 2x) \, \delta x}$ $\boxed{2}$ — 1

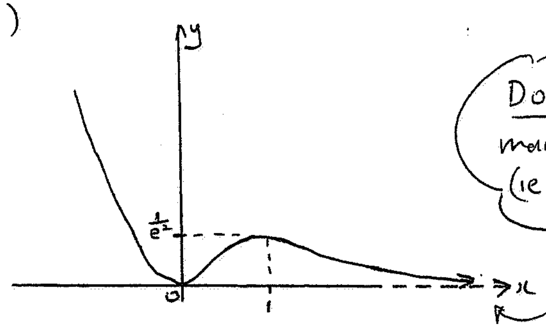
(ii) $V = \pi \int_0^2 (8 - 2x)x^2 \, dx$

$= \pi \int_0^2 (8x^2 - 2x^3) \, dx$ — 1

$= \pi \left[\dfrac{8x^3}{3} - \dfrac{x^4}{2}\right]_0^2$

$= \pi\left(\dfrac{64}{3} - 8\right) = \dfrac{40}{3}\pi \text{ units}^3$ $\boxed{2}$ — 1

(c) (i)

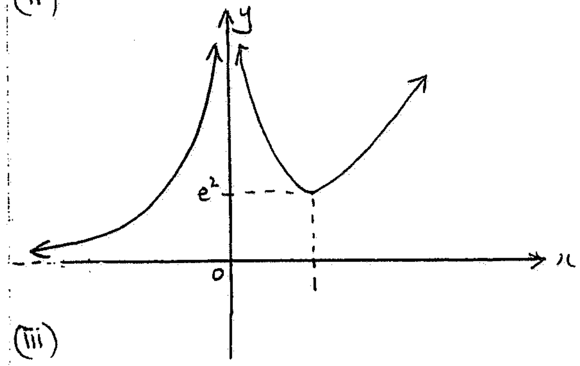


DO NOT deduct marks if the asymptotes (ie the axes) are solid. eg

$\boxed{2}$

½ asy… at… ½ sh… 1 for… at(…

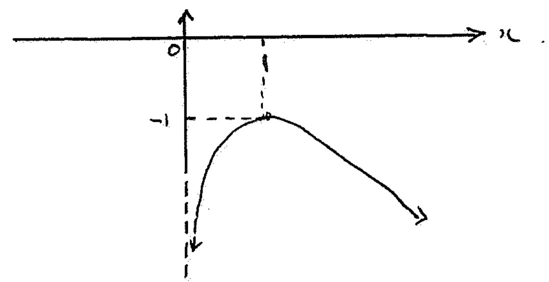
(ii)



$\boxed{2}$

½ mi… (1,… ½ sha… ½ asy… at… ½ asy… y =…

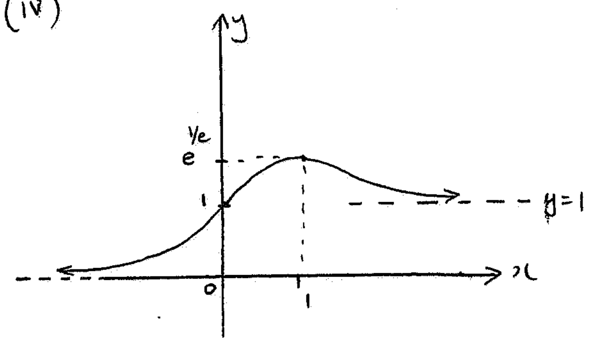
(iii)



$\boxed{2}$

1 mi… (1, −… ½ for… ½ asy…

(iv)



this asymptote must be dotted.

$\boxed{2}$

½ for (… ½ asy… at… ½ for… at(… ½ for…

Question 4 (15 Marks)

(a) $P(z) = z^7 - 1$

(i) Let $z^7 - 1 = 0$

$z^7 = 1 \Rightarrow |z| = 1$ — 1

Let $z = \cos\theta + i\sin\theta$, then

$(\cos\theta + i\sin\theta)^7 = 1$

$\cos 7\theta + i\sin 7\theta = 1$

$\therefore \cos 7\theta = 1 \quad \& \quad \sin 7\theta = 0$

$7\theta = 2k\pi$

$\theta = \frac{2k\pi}{7}$

$k=0, \quad z_0 = \text{cis } 0 = 1$

$k=1, \quad z_1 = \text{cis } \frac{2\pi}{7}$

$k=2, \quad z_2 = \text{cis } \frac{4\pi}{7}$

$k=3, \quad z_3 = \text{cis } \frac{6\pi}{7}$

$k=4, \quad z_4 = \text{cis } \frac{8\pi}{7}$

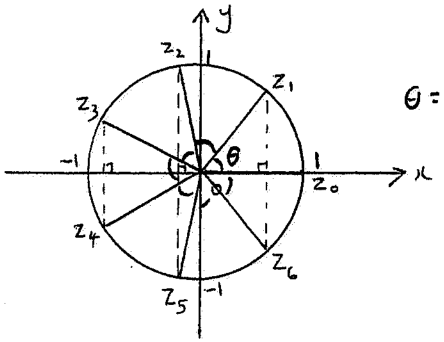
$k=5, \quad z_5 = \text{cis } \frac{10\pi}{7}$

$k=6, \quad z_6 = \text{cis } \frac{12\pi}{7}$

Alternatively, accept $z_n = \text{cis } \frac{2n\pi}{7}$ where $n = 0, 1, 2, \ldots, 6$ — 1

$\boxed{2}$

(ii)



$\theta = \frac{2\pi}{7}$

1 correct labelling

$\frac{1}{2}$ for radius

$\frac{1}{2}$ for showing symmetry.

$\boxed{2}$

NB: This doesn't need to be exact, need to show symmetry, or show the angle as $\frac{2\pi}{7}$.

(iii)

$z_0 = 1$

$z_1 = \cos\frac{2\pi}{7} + i\sin\frac{2\pi}{7}$

$z_2 = \cos\frac{4\pi}{7} + i\sin\frac{4\pi}{7}$

$z_3 = \cos\frac{6\pi}{7} + i\sin\frac{6\pi}{7}$

$z_4 = \cos\frac{8\pi}{7} + i\sin\frac{8\pi}{7} = \cos\frac{6\pi}{7} - i\sin\frac{6\pi}{7}$

$z_5 = \cos\frac{10\pi}{7} + i\sin\frac{10\pi}{7} = \cos\frac{4\pi}{7} - i\sin\frac{4\pi}{7}$

$z_6 = \cos\frac{12\pi}{7} + i\sin\frac{12\pi}{7} = \cos\frac{2\pi}{7} - i\sin\frac{2\pi}{7}$ — 1

$$P(z) = (z-1)(z-z_1)(z-z_6)(z-z_2)(z-z_5)(z-z_3)(z-z_4)$$

$$= (z-1)\left(z^2 - (z_1+z_6)z + z_1z_6\right)\left(z^2 - (z_2+z_5)z + z_2z_5\right)\left(z^2 - (z_3+z_4) + z_3z_4\right)$$

$$\therefore P(z) = (z-1)\left(z^2 - 2\cos\frac{2\pi}{7}z + 1\right)\left(z^2 - 2\cos\frac{4\pi}{7}z + 1\right)\left(z^2 - 2\cos\frac{6\pi}{7}z + 1\right)$$ — 1+1

$\boxed{3}$

(iv) Sum of all roots $= -\frac{\text{coefficient of } z^6}{\text{coefficient of } z^7}$

where $P(z) = z^7 - 1$

$\therefore z_0 + z_1 + z_2 + z_3 + z_4 + z_5 + z_6 = 0$ — 1

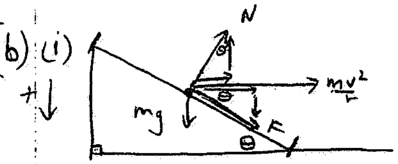
ie $1 + 2\cos\frac{2\pi}{7} + 2\cos\frac{4\pi}{7} + 2\cos\frac{6\pi}{7} = 0$ — 1

$2\left[\cos\frac{2\pi}{7} + \cos\frac{4\pi}{7} + \cos\frac{6\pi}{7}\right] = -1$

$\cos\frac{2\pi}{7} + \cos\frac{4\pi}{7} + \cos\frac{6\pi}{7} = \frac{-1}{2}$

$\boxed{2}$

(b) (i)



(i) For $V = 20$

$N\cos\theta - F\sin\theta = mg$ --- (A)

$N\sin\theta + F\cos\theta = \frac{mv^2}{r}$

$= \frac{400m}{800}$

ie $N\sin\theta + F\cos\theta = \frac{m}{2}$ --- (B) — 1

NB: if forces are resolved

(A) $\times \sin\theta$: $N\cos\theta\sin\theta - F\sin^2\theta = mg\sin\theta$ --- (C)

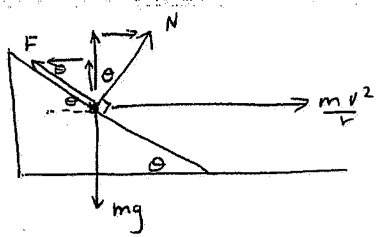
(B) $\times \cos\theta$: $N\sin\theta\cos\theta + F\cos^2\theta = \frac{m\cos\theta}{2}$ --- (D) — 1

(D) − (C): $F\cos^2\theta + F\sin^2\theta = \frac{m}{2}\cos\theta - mg\sin\theta$

$F(\cos^2\theta + \sin^2\theta) = m\left(\frac{1}{2}\cos\theta - g\sin\theta\right)$

$\boxed{2}$

(ii) (α)



$\downarrow +$

<u>For V=10</u>

$N\cos\theta + F\sin\theta = mg$ ---- (A)

$N\sin\theta - F\cos\theta = \frac{mv^2}{r}$

$= \frac{100m}{800}$

ie $N\sin\theta - F\cos\theta = \frac{m}{8}$ --- (B)

(A)$\times \sin\theta$: $N\cos\theta\sin\theta + F\sin^2\theta = mg\sin\theta$ --- (C)

(B)$\times\cos\theta$: $N\sin\theta\cos\theta - F\cos^2\theta = \frac{m\cos\theta}{8}$ --- (D)

(C) - (D) . $F(\sin^2\theta + \cos^2\theta) = m\left(g\sin\theta - \frac{\cos\theta}{8}\right)$

$F = m\left(g\sin\theta - \frac{\cos\theta}{8}\right)$ 1

Now $\cancel{m}\left(g\sin\theta - \frac{\cos\theta}{8}\right) = \cancel{m}\left(\frac{1}{2}\cos\theta - g\sin\theta\right)$

$2g\sin\theta = \frac{5}{8}\cos\theta$

$\tan\theta = \frac{5}{16g}$

$\therefore \tan\theta = \frac{5}{16\times 9.8}$ $(g = 9.8)$

$\tan\theta = 0.031887755$

$\theta \doteqdot \tan^{-1}(0.031887755..)$

$\theta \doteqdot 1°50'$ (nearest minute) 1

$\boxed{2}$

(β) From (A) & (B) and $F = 0$: $N\cos\theta = mg$ ---- (E)

$N\sin\theta = \frac{mv^2}{800}$ ---- (F) 1

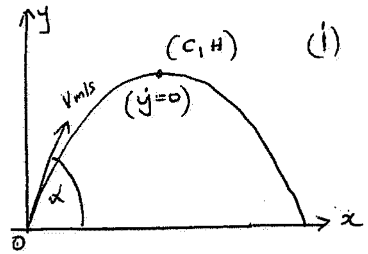
(F) $\div$ (E) $\tan\theta = \frac{v^2}{800g}$

$\frac{5}{16\times \cancel{9.8}} = \frac{v^2}{800\times\cancel{9.8}}$

$\therefore v^2 = \frac{5\times 800}{16}$ $(= 250)$

<u>Question 5</u> (<u>15 Marks</u>)

(a)



(i) For maximum height ($\dot{y} = 0$)

ie $-gt + v\sin\alpha = 0$

$t = \frac{v}{g}\sin\alpha$ 1

At $t = \frac{v}{g}\sin\alpha$, $y = H \Longrightarrow$

$\therefore H = -\frac{g}{2}\left(\frac{v}{g}\sin\alpha\right)^2 + v\sin\alpha \times \frac{v\sin\alpha}{g}$

$= \frac{-v^2}{2g}\sin^2\alpha + \frac{v^2}{g}\sin^2\alpha$

$\therefore H = \frac{v^2}{2g}\sin^2\alpha$ ----- (A) 1

also, at $t = \frac{v}{g}\sin\alpha$, $x = C \Longrightarrow$

$C = V\cos\alpha \times \frac{v}{g}\sin\alpha$

$\therefore C = \frac{v^2}{g}\sin\alpha\cos\alpha$ ---- (B) 1

From (A) & (B) : $\frac{H}{C} = \frac{\cancel{v^2}\sin^{\cancel{2}}\alpha}{2\cancel{g}} \times \frac{\cancel{g}}{\cancel{v^2}\cancel{\sin\alpha}\cos\alpha}$

$= \frac{\sin\alpha}{2\cos\alpha}$

$\frac{2H}{C} = \tan\alpha$ 1

$\alpha = \tan^{-1}\frac{2H}{C}$

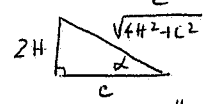
$\boxed{4}$

(ii) From (A): $H = \frac{v^2\sin^2\alpha}{2g}$

$v^2 = \frac{2gH}{\sin^2\alpha}$

$= \frac{2gH}{\left(\frac{2H}{\sqrt{4H^2+C^2}}\right)^2}$ 1

from $\tan\alpha = \frac{2H}{C}$



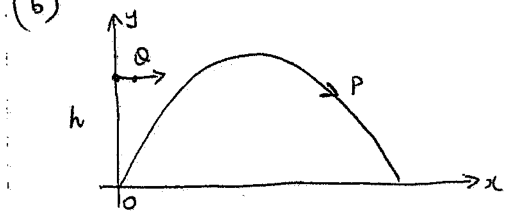
$\sin\alpha = \frac{2H}{\sqrt{4H^2+C^2}}$ 1 +

$= \frac{2gH.(4H^2+C^2)}{4H^2}$

$\therefore V^2 = \frac{g}{2H}(4H^2+C^2)$

$\boxed{3}$

(b)



$\downarrow +$

(i) For particle Q: $\ddot{x} = 0$, $\dot{x} = u$, $x = ut$; $\ddot{y} = -g$, $\dot{y} = -gt$, $y = -\frac{gt^2}{2} + h$ 1

For a collision, $v\cos\alpha = u$ (from P: $\dot{x} = v\cos\alpha$, from Q: $\dot{x} = u$)

$$\cos\alpha = \frac{u}{v}$$ 1

Now since $0 < \alpha < \frac{\pi}{2}$

$$0 < \cos\alpha < 1$$ 1

$$0 < \frac{u}{v} < 1$$

$$0 < u < v$$

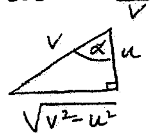
$\therefore$ $v > u$ as required $\boxed{3}$

(ii) At time of collision, y values are same:

$$-\cancel{\frac{gt^2}{2}} + h = -\cancel{\frac{gt^2}{2}} + vt\sin\alpha$$

$$t = \frac{h}{v\sin\alpha}$$ 1

$$= \frac{h}{\cancel{v}\,\frac{\sqrt{v^2-u^2}}{\cancel{v}}} \quad \text{from } \cos\alpha = \frac{u}{v}$$ 1



$$t = \frac{h}{\sqrt{v^2-u^2}}$$

$\therefore \sin\alpha = \frac{\sqrt{v^2-u^2}}{v}$

$\boxed{2}$

(iii) To collide on horizontal level, $y = 0$:

$$-\frac{gt^2}{2} + h = 0$$

$$t^2 = \frac{2h}{g}$$ 1

But $t = \frac{h}{\sqrt{v^2-u^2}}$ from (ii)

$$\therefore \frac{2h}{g} = \left[\frac{h}{\sqrt{v^2-u^2}}\right]^2$$ 1

$$\frac{2\cancel{h}}{g} = \frac{h^{\cancel{2}}}{v^2-u^2}$$

$$v^2 - u^2 = \frac{hg}{2}$$ 1

$$v^2 = \frac{1}{2}gh + u^2$$

$\boxed{3}$

Question 6 (15 marks)

(a) Since the polynomial has real coefficients and $1+2i$ is a zero of $P(x)$, then $1-2i$ is also a zero. 1

$\therefore P(x) = (x^2 - 2x + 5)(ax - 3)$, since -15 is a constant 1

$P(2) = 5 \Longrightarrow (4 - 4 + 5)(2a - 3) = 5$

$a = 2$ 1

$\therefore P(x) = (2x - 3)(x^2 - 2x + 5)$ $\boxed{4}$ 1

(b)(i) $x^3 - 4x + 5 = 0$ has roots α, β, γ.

ie
$$\alpha^3 - 4\alpha + 5 = 0$$
$$\beta^3 - 4\beta + 5 = 0$$
$$\gamma^3 - 4\gamma + 5 = 0$$

Adding these: $\alpha^3 + \beta^3 + \gamma^3 - 4(\alpha + \beta + \gamma) + 5 \times 3 = 0$ 1

$\alpha^3 + \beta^3 + \gamma^3 - 4 \times 0 + 15 = 0$ from ($\alpha + \beta + \gamma = \frac{-b}{a} = 0$)

$\therefore \alpha^3 + \beta^3 + \gamma^3 = -15$ 1

$\boxed{2}$

(ii) Since $\alpha + \beta + \gamma = 0$

$\alpha + \beta = -\gamma$, $\alpha + \gamma = -\beta$, $\gamma + \beta = -\alpha$

$(\alpha+\beta)^2(\alpha+\gamma)^2(\beta+\gamma)^2 = (-\gamma)^2(-\beta)^2(-\alpha)^2$ 1

$= \alpha^2\beta^2\gamma^2 = (\alpha\beta\gamma)^2 = 5^2 = 25$ 1

(c) (i) Let $y = x^4 - 4x^3 - 18x^2$
$y = mx + b$

$$x^4 - 4x^3 - 18x^2 = mx + b$$
$$x^4 - 4x^3 - 18x^2 - mx - b = 0$$

This has a common tangent at 2 distinct points on the curve

$\therefore$ These roots will be double roots.

Let these roots be p, p, q, q 1

(α) Now $p + p + q + q = \left(\frac{-b}{a}\right) = 4$ 1

$$2p + 2q = 4$$
$$\underline{p + q = 2}$$ $\boxed{2}$

(β) $p \times p \times q \times q = \left(\frac{e}{a}\right) = -b$ 1

$$\underline{p^2 q^2 = -b}$$ $\boxed{1}$

(ii) Sum of roots 3 at a time: $2p^2q + 2pq^2 \left(= -\frac{d}{a}\right) = m$

$$2pq(p+q) = m$$
$$2pq(2) = m$$
$$pq = \frac{m}{4}$$ 1

Now, sum of roots 2 at a time: $p^2 + q^2 + 4pq = -18$

$$(p+q)^2 + 2pq = -18$$
$$4 + \frac{m}{4} = -18$$
$$m = -44$$ 1

$$\therefore pq = \frac{m}{4}$$
$$pq = -\frac{44}{4} = -11$$
$$p^2q^2 = 121$$

but $p^2q^2 = -b$

$$\therefore b = -121$$ 1

$$\therefore \underline{y = -44x - 121}$$ 1 $\boxed{4}$

Alternatively

$$f(x) = x^4 - 4x^3 - 18x^2 - mx - b$$
$$f'(x) = 4x^3 - 12x^2 - 36x - m$$

Let roots be p, q, r

$$\therefore p + q + r = 3$$
$$2 + r = 3$$
$$\underline{r = 1}$$

also $pqr = +\frac{m}{4}$

also $pq + pr + rq = -9$

$$pq + r(p+q) = -9$$
$$pq + 1(2) = -9$$
$$pq = -11$$
$$p^2q^2 = 121$$
$$b = -121 \quad (\text{since } p^2q^2 = -b)$$

Since $pqr = \frac{m}{4}$

$$-11(1) = \frac{m}{4}$$
$$m = -44$$

$$\therefore \underline{y = -44x - 121}$$

Question 7 (15 Marks)

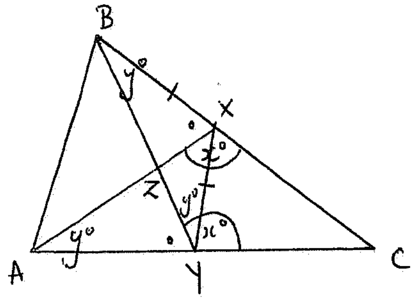
(a) (i) Choose 5, then choose any of remaining, then choose match:

$\therefore P(\text{total } 15) = \frac{1}{1} \times \frac{8}{8} \times \frac{1}{7}$ $\boxed{1}$ 1

(ii) Choose 2, then choose one of 4, 5, 6, 7, 8, 9, then choose match:

$\therefore P(\text{total } 15) = \frac{1}{1} \times \frac{6}{8} \times \frac{1}{7} = \frac{3}{28}$ 1 for 6/8, 1 for ans

(b)



$\angle BXA = \angle AYB = 180° - x$ (angle sum of straight angle is 180°) 1

$\therefore ABXY$ is a cyclic quadrilateral (equal angles subtended to the same side from the line AB) 1 $\boxed{2}$

$\angle XBY = \angle XYB = y°$ (angles opposite equal sides BX & YX) 1

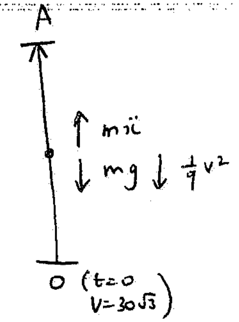
$\angle XBY = \angle XAY = y°$ (angles to the circumference of a circle subtended from the same segment XY, are equal.) 1

$\angle BAX = \angle BYX = y°$ (angles to the circumference of a circle subtended from the same segment BX are equal) Same reason as above.

$\therefore \angle BAX = \angle XAY = y°$

$\therefore AX$ bisects $\angle BAC$ $\boxed{2}$

(c)



(i) ↑+

$m\ddot{x} = -mg - \frac{1}{9}v^2$

$10\ddot{x} = -100 - \frac{1}{9}v^2$

$\ddot{x} = -10 - \frac{1}{90}v^2$ $\boxed{1}$ 1

(ii) $\ddot{x} = -10 - \frac{1}{90}v^2$

$\frac{dv}{dt} = \frac{-900 - v^2}{90}$

$\frac{dt}{dv} = -\frac{90}{900+v^2}$

$t = -90\int \frac{1}{900+v^2}\,dv$

$t = -\frac{90}{30}\tan^{-1}\frac{v}{30} + c$

at $t = 0$, $v = 30\sqrt{3}$:

$0 = -3\tan^{-1}\frac{30\sqrt{3}}{30} + c$

$c = \pi$ 1

$\therefore t = \pi - 3\tan^{-1}\frac{v}{30}$

at A, $v = 0 \Rightarrow$ $t = \pi - 3\tan^{-1} 0$

$t = \pi$ seconds $\boxed{2}$ 1

d) (i) Show $\cot 2x - \tan 2x = 2\cot 4x$

$$\begin{aligned}
LHS &= \cot 2x - \tan 2x \\
&= \frac{1}{\tan 2x} - \tan 2x \\
&= \frac{1-\tan^2 2x}{\tan 2x} \\
&= 2\left(\frac{1-\tan^2 2x}{2\tan 2x}\right) \\
&= 2\left(\frac{1}{\tan 4x}\right) \\
&= 2\cot 4x \\
&= RHS
\end{aligned}$$

1

1

$\therefore LHS = RHS$ $\boxed{2}$

(ii) Prove by induction that

$\tan x + 2\tan 2x + 4\tan 4x + \dots + 2^{n-1}\tan(2^{n-1}x) = \cot x - 2^n\cot(2^n x)$.

Step 1: Show true for $n=1$

ie Show $\tan x = \cot x - 2\cot 2x$

$$\begin{aligned}
RHS &= \cot x - 2\cot 2x \\
&= \frac{1}{\tan x} - \cancel{2}\left(\frac{1-\tan^2 x}{\cancel{2}\tan x}\right) \\
&= \frac{\cancel{1} - \cancel{1} + \tan^2 x}{\cancel{\tan x}} \\
&= \tan x \\
&= LHS
\end{aligned}$$

1

Step 2: Assume true for $n=k$

ie Assume $\tan x + 2\tan 2x + \dots + 2^{k-1}\tan(2^{k-1}x) = \cot x - 2^k\cot(2^k x)$

Show true for $n = k+1$

ie show $\tan x + 2\tan 2x + \dots + 2^{k-1}\tan(2^{k-1}x) + 2^k\tan 2^k x = \cot x - 2^{k+1}\cot(2^{k+1}x)$.

$$\begin{aligned}
\therefore LHS &= \cot x - 2^k\cot(2^k x) + 2^k\tan 2^k x \quad \text{(by assumption)} \\
&= \cot x - 2^k\left[\cot(2^k x) + \tan(2^k x)\right] \\
&= \cot x - 2^k\left[2\cot(2.2^k x)\right] \quad \text{from part (i) \& 2 replaced with } 2^k. \\
&= \cot x - 2^{k+1}\left[\cot(2^{k+1}x)\right]. \\
&= RHS
\end{aligned}$$

1

1

Question 8 (15 Marks)

(b.) (i) $xy = c^2$

$$\begin{aligned}
y &= c^2x^{-1} \\
\frac{dy}{dx} &= -c^2x^{-2}
\end{aligned}$$

at $x = cp$, $\frac{dy}{dx} = \frac{-\cancel{c^2}}{\cancel{c^2}p^2} = \frac{-1}{p^2}$ ← gradient of tangent

$\therefore$ Gradient of normal $= p^2$

1

Equation of normal ⟹

$$\begin{aligned}
y - \frac{c}{p} &= p^2(x - cp) \\
yp - c &= p^3x - cp^4 \\
p^3x - py &= c(p^4-1)
\end{aligned}$$

1 $\boxed{2}$

(ii) Since $Q\left(cq, \frac{c}{q}\right)$ lies on normal in (i), it satisfies the eq^n,

$$\begin{aligned}
p^3.cq - p.\frac{c}{q} &= c(p^4-1) \\
p^3q - \frac{p}{q} &= p^4 - 1 \\
p^4 - p^3q + \frac{p}{q} - 1 &= 0 \\
p^3(p-q) + \frac{1}{q}(p-q) &= 0 \\
(p-q)\left(p^3 + \frac{1}{q}\right) &= 0
\end{aligned}$$

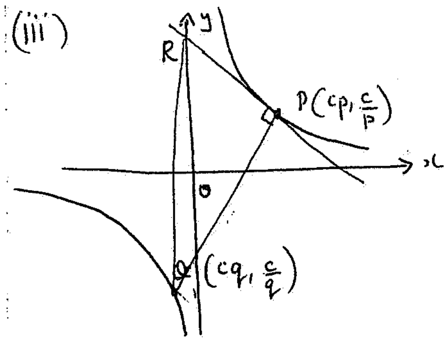
1

$p^3 + \frac{1}{q} = 0$ only as $p \neq q$.

$\underline{p^3q = -1}$

1 $\boxed{2}$

(iii)



Eq^n of tangent is given by:

$x + p^2y = 2cp$

y intercept $(x=0)$:

$$\begin{aligned}
p^2y &= 2cp \\
y &= \frac{2c}{p}
\end{aligned}$$

$\therefore R\left(0, \frac{2c}{p}\right)$

Now $Q\left(cq, \frac{c}{q}\right)$, now from (ii) $p^3 q = -1$, $q = \frac{-1}{p^3}$

$\Rightarrow Q$ is $\left(\frac{-c}{p^3}, -cp^3\right)$

Area of $\Delta PQR = \frac{1}{2} \times PR \times PQ$ ---- ①

PR = perpendicular distance from $R\left(0, \frac{2c}{p}\right)$ to the normal $p^3 x - py - c(p^4 - 1) = 0$

$$= \frac{\left|-p \times \frac{2c}{p} - c(p^4-1)\right|}{\sqrt{(p^3)^2 + (-p)^2}}$$

$$= \frac{\left|-2c - c(p^4-1)\right|}{|p|\sqrt{p^4+1}}$$

$$= \frac{c\left|-1-p^4\right|}{|p|\sqrt{p^4+1}}$$

$$= \frac{c\left(1+p^4\right)}{|p|\sqrt{p^4+1}}$$

PQ = perpendicular distance from $Q\left(\frac{-c}{p^3}, -cp^3\right)$ to the tangent $x + p^2 y - 2cp = 0$

$$= \frac{\left|\frac{-c}{p^3} + p^2(-cp^3) - 2cp\right|}{\sqrt{1^2 + (p^2)^2}}$$

$$= \frac{\left|\frac{-c}{p^3} - cp^5 - 2cp\right|}{\sqrt{1+p^4}}$$

$$= \frac{\frac{c}{|p|^3}\left|-1-p^8-2p^4\right|}{\sqrt{1+p^4}}$$

$$= \frac{c}{|p|^3}\frac{\left|1+2p^4+p^8\right|}{\sqrt{1+p^4}}$$

$$= \frac{c}{|p|^3}\frac{(p^4+1)^2}{\sqrt{1+p^4}}$$

NB: The above method uses the ⊥ distance from a point to a line. Alternatively, they may find PR & PQ using the distance rule.

Alternatively: Using the distance Rule to find PR & PQ.

$$PR = \sqrt{\left(\frac{c}{p} - \frac{2c}{p}\right)^2 + (cp)^2} = \sqrt{c^2\left(\frac{1}{p^2} + p^2\right)} = \frac{c}{|p|}\sqrt{1+p^4}$$

$$PQ = \sqrt{(cq-cp)^2 + \left(\frac{c}{q} - \frac{c}{p}\right)^2} = \sqrt{c^2(q-p)^2 + c^2\left(\frac{p-q}{p^2q^2}\right)^2}$$

$$= c\sqrt{(q-p)^2 + \frac{(p-q)^2}{p^2q^2}} = c|p-q|\sqrt{1 + \frac{1}{p^2 \cdot \frac{1}{p^6}}}$$ (from $q = \frac{-1}{p^3}$

$$= c\left|p + \frac{1}{p^3}\right|\sqrt{1+p^4}$$

$$= c\frac{(p^4+1)}{|p|^3} \cdot \sqrt{1+p^4}$$

$\therefore$ Area $\Delta PQR = \frac{1}{2} \times PR \times PQ$

$$= \frac{1}{2}\frac{c^2}{|p|}\sqrt{p^4+1} \cdot \frac{(p^4+1)}{|p|^3}\sqrt{p^4+1}$$

$$= \frac{c^2}{2p^4}(p^4+1)^2$$

From ①:

$$\text{Area } \Delta PQR = \frac{1}{2} \times PR \times PQ$$

$$= \frac{1}{2} \times \frac{c}{|p|}\frac{(1+p^4)}{\sqrt{p^4+1}} \times \frac{c}{|p|^3}\frac{(p^4+1)^2}{\sqrt{1+p^4}} \qquad 1$$

$$= \frac{c^2}{2p^4}\frac{(p^4+1)^3}{p^4+1}$$

$$= \frac{c^2}{2p^4}(p^4+1)^2$$

$$= \frac{c^2}{2}\left(\frac{p^4+1}{p^2}\right)^2$$

$$= \frac{c^2}{2}\left(p^2+\frac{1}{p^2}\right)^2 \qquad \boxed{2} \quad 1$$

(iv) Since $p^2+\frac{1}{p^2} = \left(p-\frac{1}{p}\right)^2+2$, it has a minimum value of 2 when $\underline{p=1 \text{ or } -1}$ 1 1 $\boxed{2}$

(b) (i) $LHS = (1+x)^{2n} + (1-x)^{2n}$

$$= {}^{2n}C_0 + \cancel{{}^{2n}C_1 x} + {}^{2n}C_2 x^2 + \dots + {}^{2n}C_{2n-1}x^{2n-1} + \cancel{{}^{2n}C_{2n}x^{2n}}$$
$$\quad {}^{2n}C_0 - \cancel{{}^{2n}C_1 x} + {}^{2n}C_2 x^2 + \dots + {}^{2n}C_{2n-1}x^{2n-1} - \cancel{{}^{2n}C_{2n}x^{2n}} \qquad 1$$

$$= 2\left[{}^{2n}C_0 + {}^{2n}C_2 x^2 + {}^{2n}C_4 x^4 + \dots\right] \qquad 1$$

$$= 2\left[\sum_{r=0}^{n} {}^{2n}C_{2r}\, x^{2r}\right] \quad \text{for } n=1,2,3,\dots$$

$= RHS$ $\boxed{2}$

(ii) A, B or C   5 letters of which 2 are A's:

2 A's	C, C, C	$=\frac{5!}{2!\,3!} = 10$
2 A's	B, B, C	$\frac{5!}{2!\,2!} = 30$
2 A's	C, C, B	$\frac{5!}{2!\,2!} = 30$
2 A's	B, B, B	$\frac{5!}{2!\,3!} = 10$

1

Total $= 10+10+30+30 = \underline{80}$.

Alt: Choose position of A's in ${}^5C_2$ ways. Each of the remaining positions can be in 2 ways $\Rightarrow 2^3$

Now ${}^5C_2 \times 2^3 = \frac{5\times 4}{2\times 1}\times 2^3 = \underline{80}$ 1

∴ Result holds true. $\boxed{2}$

(β)

No. of A's	$n(E)$
0	${}^{2n}C_0 \cdot 2^{2n-0}$
2	${}^{2n}C_2 \cdot 2^{2n-2}$
4	${}^{2n}C_4 \cdot 2^{2n-4}$
⋮	⋮
2n	${}^{2n}C_{2n} \cdot 2^{2n-2n}$

$$\text{Total} = \sum_{r=0}^{n} {}^{2n}C_{2r} \cdot 2^{2n-2r} \qquad 1$$

$$= \sum_{r=0}^{n} {}^{2n}C_{2r} \cdot 2^{-2r} \cdot 2^{2n}$$

$$= 2^{2n}\sum_{r=0}^{n} {}^{2n}C_{2r}\left(\frac{1}{2}\right)^{2r}$$

$$= 2^{2n} \times \frac{1}{2}\left[\left(1+\frac{1}{2}\right)^{2n} + \left(1-\frac{1}{2}\right)^{2n}\right] \qquad 1$$

$$= 2^{2n-1}\left[\left(\frac{3}{2}\right)^{2n} + \left(\frac{1}{2}\right)^{2n}\right]$$

$$= \frac{2^{2n-1}}{2^{2n}}\left[3^{2n}+1\right] \qquad 1$$

$$= 2^{-1}\left[3^{2n}+1\right]$$

as required. $\boxed{3}$